

*Atti Semin. Mat. Fis. Univ. Modena Reggio Emilia*, 56 (2008-2009), 95-111.

## AN ALTERNATIVE FORM OF THE FUNCTIONAL EQUATION FOR RIEMANN'S ZETA FUNCTION

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**Abstract.** In this paper we present a simple method for deriving an alternative form of the functional equation for Riemann's Zeta function. The connections between some functional equations obtained implicitly by Leonhard Euler in his work "*Remarques sur un beau rapport entre les series des puissances tant directes que reciproques*" in *Memoires de l'Academie des Sciences de Berlin* 17, (1768), permit to define a special function, named  $A(s)$ , which is fully symmetric and is similar to Riemann's  $\xi$  function<sup>1</sup>. To be complete we find several integral representations of the  $A(s)$  function and as a direct consequence of the second integral representation we obtain also an analytic continuation of the same function using an identity of Ramanujan.

2000 *Mathematics Subject Classification*: Primary 11M35; Secondary 11B68, 11M06.

Keywords: Riemann Zeta, Dirichlet Beta, Riemann Hypothesis, series representations.

### 1. INTRODUCTION

Formulae (1) and (2) below can be found in the chapter devoted to Euler's Gamma function in [4].

These are namely two functional equations for the Eulerian Zeta and for the alternating Zeta, connected with the odd numbers, best known as Dirichlet's Beta function and Catalan's Beta function, see [4, pag. 35, formulae (24) and (29)].

Both of them were discovered, over 100 years before G.F.B. Riemann and O. Schlömilch [8, notes on chapter II], by L. Euler in 1749 and published in 1768 in "*Memoires de l'Academie des Sciences de Berlin* 17", with the title of "*Remarques sur un beau rapport entre les series des puissances tant directes que reciproques*"<sup>2</sup>.

<sup>1</sup>  $\xi(s) = \prod (s/2) (s-1) \pi^{-s/2} \zeta(s)$

<sup>2</sup> Leonhardi Euleri, *Opera Omnia*: Series 1, Volume 15, pp. 70 - 90

The former gives, actually, an analytic extension to the complex half-plane with  $\Re(s) < 1$ :

$$(1) \quad \zeta(1-s) = 2(2\pi)^{-s} \Gamma(s) \cos\left(\frac{\pi s}{2}\right) \zeta(s).$$

Here,  $\Gamma$  denotes Euler's Gamma function.

Let us remember that the Riemann Zeta function  $\zeta(s)$  is defined by [15, pp. 96-97, see Section 2.3]:

$$\zeta(s) := \begin{cases} \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1-2^{-s}} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} & (\Re(s) > 1) \\ (1-2^{1-s})^{-1} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} & (\Re(s) > 0; s \neq 1) \end{cases}$$

which can indeed be analytically continued to the whole complex  $s$  plane except for a simple pole at  $s = 1$  with residue 1.

The Riemann Zeta function  $\zeta(s)$  plays a central role in the applications of complex analysis to number theory.

The number-theoretic properties of  $\zeta(s)$  are exhibited by the following result as *Euler's product formula*, which gives a relationship between the set of primes and the set of positive integers:

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1} \quad (\Re(s) > 1),$$

where the product is taken over all primes.

It is an analytic version of the fundamental theorem of arithmetic, which states that every integer can be factored into primes in an essentially unique way.

Euler used this product to prove that the sum of the reciprocals of the primes diverges.

The latter, on the contrary, gives an analytic extension on the whole complex plane of the  $L(s)$  function, that is Dirichlet's  $L$  function for the nontrivial character modulo 4, which was later denoted by  $L(s, \chi_4)$  by other authors:

$$(2) \quad L(1-s) = \left(\frac{2}{\pi}\right)^s \Gamma(s) \sin\left(\frac{\pi s}{2}\right) L(s).$$

The  $L(s)$  function was defined and used by Euler, practically for  $\Re(s) > 0$  with the expression:

$$L(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}.$$

It does not possess any singular point.

The  $L(s)$  function is also connected to the theory of primes [5] which may perhaps be best summarized by

$$L(s) = \prod_{p \equiv 1 \pmod{4}} (1 - p^{-s})^{-1} \cdot \prod_{p \equiv 3 \pmod{4}} (1 - p^{-s})^{-1} = \prod_{p \text{ odd}} \left(1 - (-1)^{\frac{p-1}{2}} p^{-s}\right)^{-1},$$

where the products are taken over primes and the rearrangement of factors is permitted because of an absolute convergence.

Among the properties of these functions, we will limit ourselves to report the following integral representations [4, pp. 32 and 35, formulae (4),(5) and (28)]:

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t - 1} dt \quad (\Re(s) > 1)$$

$$\zeta(s) = \frac{1}{(1 - 2^{1-s}) \Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t + 1} dt \quad (\Re(s) > 0)$$

$$L(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t + e^{-t}} dt \quad (\Re(s) > 0).$$

The first integral representation, due to Abel [1], is the key ingredient of the first proof of Bernhard Riemann [14], that is the proof of his classical functional equation for  $\zeta(s)$ .

Riemann obtained it by making the change of variable  $x = tn$  in the definition (via the integral) of the Gamma function, also called Eulerian integral of second kind, and then summing for all  $n \geq 1$ , as shown by the following sequence of formulae, for  $\Re(s) > 1$ :

$$\Gamma(s) = \int_0^{\infty} \frac{x^{s-1}}{e^x} dx, \quad \frac{\Gamma(s)}{n^s} = \int_0^{\infty} \frac{t^{s-1}}{e^{tn}} dt, \quad \text{and} \quad \Gamma(s) \zeta(s) = \int_0^{\infty} \frac{t^{s-1}}{e^t - 1} dt.$$

## 2. THE FUNCTIONAL EQUATION FOR THE RIEMANN ZETA FUNCTION

The functional equation for the Riemann Zeta function is shown in the asymmetrical formulation:

$$\zeta(s) = 2^s \pi^{s-1} \Gamma(1-s) \sin\left(\frac{s\pi}{2}\right) \zeta(1-s)$$

or in the symmetrical formulation:

$$\Gamma\left(\frac{s}{2}\right) \pi^{-s/2} \zeta(s) = \Gamma\left(\frac{1-s}{2}\right) \pi^{-(1-s)/2} \zeta(1-s)$$

Euler verified this relation exactly for all integer values of  $s$  and numerically to great accuracy for many fractional values as well.

Naturally Euler worked for integer values of  $s$  with what we now call Abel Summation (see [8] and also [18]).

If  $\sum_{n=0}^{\infty} a_n$  is a series such that  $\sum_{n=0}^{\infty} a_n z^n$  converges inside the unit disk, we shall say that  $\sum_{n=0}^{\infty} a_n$  is *Abel summable* to the value  $s$  if

$$\lim_{0 < x < 1, x \rightarrow 1^-} \sum_{n=0}^{\infty} a_n x^n = s$$

In particular, if the sum  $f(z)$  of the power series extends analytically to a domain containing  $z = 1$ , we can take  $f(1)$  as value of the sum: this is what Euler did.

Practically Euler proved, for any integer  $m \geq 2$ , that

$$\lim_{x \rightarrow 1^-} \frac{\eta(1-m, x)}{\eta(m, x)} := \begin{cases} -\frac{(2^m-1)}{\pi^m(2^{m-1}-1)} \Gamma(m) \cos\left(\frac{\pi m}{2}\right) & \text{if } m \text{ is even,} \\ 0 & \text{if } m \text{ is odd} \end{cases}$$

where the associated power series is  $\eta(m, x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^m} x^n$ .

He formally replaced  $\lim_{x \rightarrow 1^-} \eta(m, x)$  by  $\eta(m)$ , that is  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^m} = (1 - 2^{1-m}) \zeta(m)$ .

However he did not know how to prove this intriguing assertion for all real numbers and in 1859 Riemann was the first to indicate that the functional equation for  $\zeta(s)$  is true and more precisely he provided two different proofs of his classical functional equation [14].

Riemann is also the first to use two of the basic identities of the Gamma function, that are Euler's complement formula and Legendre's duplication formula (the explicit formulae are given below), to rewrite the asymmetrical formulation in the symmetrical formulation [14].

Nowadays, many proofs of this important result exist.

Later, mathematicians like Hardy, Siegel and others enriched the list of proofs.

For example in the classical treatise of Titchmarsh [17, pp. 16-27] seven different methods are presented .

Many proofs are based on Poisson's summation formula and other proofs are unnecessarily long or conceptually difficult.

Recently, two authors [12] have presented a short proof of Riemann's functional equation, based upon Poisson summation.

This elegant and powerful technique was used to derive a new and simple proof of Lipschitz summation formula and as a direct consequence of this the authors obtained an easy proof of the functional equation for  $\zeta(s)$ .

The main point of the paper is that Hurwitz's relation [15, pp. 89-90, formulae (6) and (7)]:

$$\zeta(1-s, a) = \frac{\Gamma(s)}{(2\pi)^s} \left\{ e^{-\pi i s/2} F(s, a) + e^{\pi i s/2} F(s, -a) \right\}$$

can be obtained as a conceptually simple corollary of Lipschitz summation.

Notice that  $F(s, a)$ , which is often referred to as the 'periodic (or Lerch) Zeta function', is defined by the following relation:

$$F(s, a) = \sum_{n=1}^{\infty} \frac{e^{2\pi i n a}}{n^s}$$

and that when  $a = 1$  in Hurwitz's relation,  $F(s, 1) = F(s, -1) = \zeta(s)$ , so that Riemann functional equation (1) follows directly from that by using  $e^{-\frac{\pi i s}{2}} + e^{\frac{\pi i s}{2}} = 2 \cos\left(\frac{\pi s}{2}\right)$ .

Here we briefly present one proof of the functional equation extracted from [17] and due to Hardy and we will see that the functional equation is strongly related to Fourier's series.

The starting point of Hardy is not the function  $\zeta(s)$ , but the function:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = \left(1 - 2^{1-s}\right) \zeta(s).$$

This Dirichlet series is convergent for all positive values of  $s$ , and so, by a general theorem on the convergence of Dirichlet series, it is convergent for all values of  $s$ , with  $\Re(s) > 0$ .

Here the pole of  $\zeta(s)$  at  $s = 1$  is cancelled by the zero of the other factor.

Hardy's proof runs as follows. Let

$$f(x) = \sum_{n=0}^{\infty} \frac{\sin(2n+1)x}{2n+1}.$$

This series is boundedly convergent and

$$f(x) = (-1)^m \frac{1}{4}\pi \quad \text{for} \quad m\pi < x < (m+1)\pi \quad (m = 0, 1, 2, \dots).$$

Multiplying by  $x^{s-1}$  ( $0 < s < 1$ ), and integrating over  $(0, \infty)$ , we obtain

$$\begin{aligned} \frac{1}{4}\pi \sum_{m=0}^{\infty} (-1)^m \int_{m\pi}^{(m+1)\pi} x^{s-1} dx &= \Gamma(s) \sin\left(\frac{s\pi}{2}\right) \sum_{n=0}^{\infty} \frac{1}{(2n+1)^{s+1}} = \\ &= \Gamma(s) \sin\left(\frac{s\pi}{2}\right) (1 - 2^{-s-1}) \zeta(s+1). \end{aligned}$$

The term-by-term integration over any finite range is permissible since the series  $f(x)$  is boundedly convergent.

The series on the left hand side is

$$\frac{\pi^s}{s} \left[ 1 + \sum_{m=1}^{\infty} (-1)^m \{(m+1)^s - m^s\} \right].$$

This series is convergent for  $s < 1$  and also uniformly convergent for  $\Re(s) < 1$ .

Its sum is therefore an analytic function of  $s$ , regular for  $\Re(s) < 1$ .

But for  $s < 0$  it is  $2(1^s - 2^s + 3^s - \dots) = 2(1 - 2^{s+1}) \zeta(-s)$ .

Its sum is therefore the same analytic function of  $s$  for  $\Re(s) < 1$ .

Hence, for  $0 < s < 1$ ,

$$\frac{\pi^{s+1}}{2s} (1 - 2^{s+1}) \zeta(-s) = \Gamma(s) \sin\left(\frac{s\pi}{2}\right) (1 - 2^{-s-1}) \zeta(s+1),$$

and the functional equation again follows.

We close this section, indicating that in the Appendix of this paper we provide an application of Poisson's summation formula.

3. THE SPECIAL FUNCTION  $A(s)$ 

At this stage, let us introduce a special function defined (in a somewhat inelegant manner) by the following symbolic relation:

$$(3) \quad A(1-s) = \frac{\Gamma(1-s) \zeta(1-s) L(1-s)}{\pi^{1-s}}.$$

Changing (1) and (2) in (3), and considering Euler's complement formula, that is true for the identification principle of the relations among analytic functions, on the whole complex plane, except for the integer values:  $s = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 7, \dots$ , that is:

$$\Gamma(s) \cdot \Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$

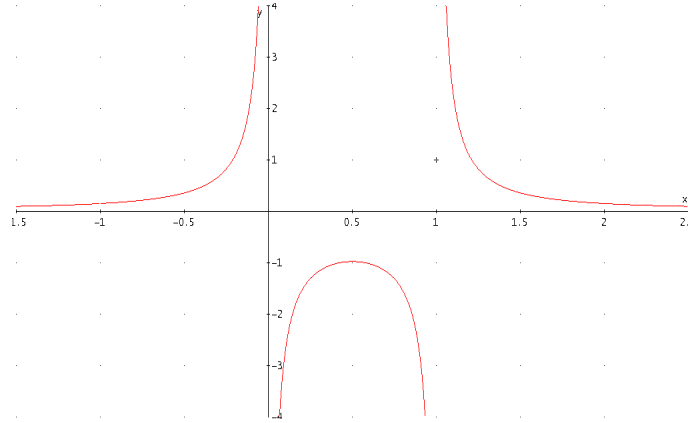
we have:

$$(4) \quad A(1-s) = \frac{\Gamma(s) \zeta(s) L(s)}{\pi^s} = A(s).$$

The  $A(s)$  function is actually a meromorphic function, that satisfies the following remarkable identity:

$$(5) \quad A(s) = A(1-s).$$

Such an identity shows a symmetry of the  $A(s)$  function with respect to the vertical straight line  $\Re(s) = 1/2$  [in particular let us consider the plot produced in the interval  $0 < \Re(s) < 1$  by the software product *DERIVE*<sup>3</sup> *Version 6.1* for Windows, for the real part of the analytic function  $A(s)$  ( see the curve in Fig. 1 )].



**Fig. 1**

<sup>3</sup> *DERIVE* is a Computer Algebra System distributed by Texas Instruments.

The identity (5) was obtained in a functional manner and leads to a functional equation stemming from the entire function:

$$(6) \quad \frac{\Gamma(s) \zeta(s) L(s)}{\Gamma(1-s) \zeta(1-s) L(1-s)} = \frac{\pi^s}{\pi^{1-s}} = \exp[(2s-1) \log \pi].$$

To verify this, we proceed as follows.

From (6) we get the ratio between  $\zeta(s)$  and  $\zeta(1-s)$ :

$$\frac{\zeta(s)}{\zeta(1-s)} = \frac{\pi^s \Gamma(1-s) L(1-s)}{\pi^{1-s} \Gamma(s) L(s)}.$$

From this, re-using (2), we immediately get Riemann's well known functional equation [4, pag. 35, formula (23)]:

$$(7) \quad \zeta(s) = 2^s \pi^{s-1} \Gamma(1-s) \sin\left(\frac{\pi s}{2}\right) \zeta(1-s).$$

In short, we have rewritten the following functional identity:

$$(8) \quad \Gamma(s) \pi^{-s} \zeta(s) L(s) = \Gamma(1-s) \pi^{-(1-s)} \zeta(1-s) L(1-s)$$

in the form (7) and in this case we have not used Euler's complement formula of the function  $\Gamma(s)$ , but we have used only the functional equation of  $L(s)$ .

In Riemann's memoir the functional identity is in the form<sup>4</sup>:

$$\prod \left(\frac{s}{2} - 1\right) \pi^{-s/2} \zeta(s) = \prod \left(\frac{1-s}{2} - 1\right) \pi^{-(1-s)/2} \zeta(1-s)$$

that is slightly different from (8).

This approach is the motivation for the title of paper : "An alternative form of the functional equation for Riemann's Zeta function".

#### 4. AN INTEGRAL REPRESENTATION

By using the identities [3, chap. X, p. 355, 10.15] :

$$(9) \quad \Gamma(s) a^{-s} = \int_0^\infty x^{s-1} e^{-ax} dx \equiv M_s \{e^{-ax}\}$$

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<sup>4</sup>  $\prod(s)$  is related to the standard Gamma function introduced by *Legendre* by the equation  $\prod(s) = \Gamma(s+1)$ .



where  $M_s$  denotes the Mellin transform and

$$(10) \quad \sum_{\mathfrak{m}} (-1)^{\mathfrak{m}-1} e^{-\mathfrak{m}^2 x} = \frac{1}{2} [1 - \theta_4(0 | ix/\pi)]$$

where  $\theta_4(z|\tau)$  is the well-known  $\theta(z|\tau)$  theta function of Jacobi [19, chap. XXI] and the summation variable  $\mathfrak{m}$  is to run over all positive integers, we derive the following integral representation of the  $A(s)$  function:

$$(11) \quad A(s) = \frac{\pi^{-s}}{1-2^{1-s}} \int_0^\infty x^{s-1} \left\{ \frac{1}{4} [1 - \theta_4^2(0 | ix/\pi)] \right\} dx.$$

The key to obtain the previous formula are the works [6, pp. 409-410] and [7] by M.L. Glasser.

Combining the following two Mellin transforms:

$$(1 - 2^{1-2s}) [\Gamma(s) \zeta(2s)] = M_s \left\{ \frac{1}{2} [1 - \theta_4(0 | ix/\pi)] \right\}, \quad \Re(s) > 0$$

and <sup>5</sup>

$$(1 - 2^{1-2s}) [\Gamma(s) \zeta(2s)] - (1 - 2^{1-s}) [\Gamma(s) \zeta(s) L(s)] = M_s \left\{ \frac{1}{4} [1 - \theta_4(0 | ix/\pi)]^2 \right\}, \quad \Re(s) > 0$$

the former is immediately obtained from Eqs. (9) and (10) and the latter is obtained integrating term by term the following remarkable identity, obtained from an identity<sup>6</sup> by Jacobi and the result  $\theta_4^2(0|\tau) = 2k'K/\pi$  ([19], p. 479):

$$\frac{1}{4} [1 - \theta_4^2(0 | ix/\pi)] = \sum_{\mathfrak{l}} (-1)^{(\mathfrak{l}-1)/2} [e^{\mathfrak{l}x} + 1]^{-1}$$

(here the sum is to be expanded as a geometric series in  $e^{-\mathfrak{l}x}$ :

$$e^{-\mathfrak{l}x} - e^{-2\mathfrak{l}x} + e^{-3\mathfrak{l}x} - e^{-4\mathfrak{l}x} + e^{-5\mathfrak{l}x} - \dots = [e^{\mathfrak{l}x} + 1]^{-1}$$

and the summation variable  $\mathfrak{l}$  is to run over all positive odd integers), we are in the position to determine the integral representation (11) for the  $A(s)$  function, by the linearity property of the Mellin transformation, from the following remarkable identity:

$$A(s) = \frac{\Gamma(s) \zeta(s) L(s)}{\pi^s} \equiv \frac{\pi^{-s}}{1-2^{1-s}} \left[ M_s \left\{ \frac{1}{2} [1 - \theta_4(0 | ix/\pi)] \right\} - M_s \left\{ \frac{1}{4} [1 - \theta_4(0 | ix/\pi)]^2 \right\} \right]$$

<sup>5</sup> See the correct formula in "Solving some problems of advanced analytical nature posed in the SIAM-review", by C.C. Grosjean, pag 432, Bull. Belg. Math. Soc. **3** (1996) and also in [7].

<sup>6</sup> See several identities discovered by Jacobi [10].

$$= \frac{\pi^{-s}}{1-2^{1-s}} M_s \left\{ \frac{1}{4} \left[ 1 - \theta_4^2(0 | ix/\pi) \right] \right\}.$$

The integral representation (11) is valid for  $\Re(s) > 0$ , in fact we can see that it contains the product  $(1-2^{1-s}) \zeta(s)$  and it gives origin to Euler's function  $\eta(s)$  [known as Dirichlet's Eta function], which is defined just for  $\Re(s) > 0$  through the following alternating series:

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n)^s}.$$

## 5. THE A(s) FUNCTION AND THE HARMONIC SUM

Define

$$(12) \quad H(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} (-1)^m e^{-n(2m+1)x}.$$

This sum is also called the “harmonic sum”.

Applying the basic functional properties of the Mellin transform ([11], see APPENDIX), we find:

$$(13) \quad M_s \{H(x)\} = \int_0^{\infty} H(x) x^{s-1} dx = \sum_{n=1, m=0}^{\infty} (-1)^m \int_0^{\infty} e^{-n(2m+1)x} x^{s-1} dx =$$

$$\sum_{n=1, m=0}^{\infty} (-1)^m n^{-s} (2m+1)^{-s} \int_0^{\infty} e^{-t} t^{s-1} dt = \sum_{n=1}^{\infty} n^{-s} \sum_{m=0}^{\infty} (-1)^m (2m+1)^{-s} \int_0^{\infty} e^{-t} t^{s-1} dt =$$

$$\zeta(s) L(s) \Gamma(s).$$

The interchange of summation and integration is legitimate by Fubini's theorem.

Now we observe that:

$$\sum_{n=1}^{\infty} \sum_{m=0}^{\infty} (-1)^m e^{-n(2m+1)x} = \sum_{n=1}^{\infty} \frac{e^{-nx}}{1 + e^{-2nx}}$$

and besides that:

$$H(x) = \sum_{n=1}^{\infty} \frac{e^{-nx}}{1 + e^{-2nx}} \leq \sum_{n=1}^{\infty} e^{-nx} = e^{-x} / (1 - e^{-x}).$$

This result establishes that the fundamental strip of the Mellin transform is  $(1, +\infty)$  and that (13) exits for any complex number  $s > 1$ .

From (13) we obtain another integral representation for the  $A(s)$  function, that is :

$$(14) \quad A(s) = \frac{1}{\pi^s} \int_0^{\infty} H(x) x^{s-1} dx \quad \Re(s) > 1$$

where  $H(x)$  is the function defined by the infinite harmonic sum (12).

Now we use (14) to give an independent proof of (5) that does not use (1) and (2).

We first consider the following transformation of  $H(x)$ , that is:

$$(15) \quad H(x) = \frac{\pi}{4x} - \frac{1}{4} + \frac{\pi}{x} H\left(\frac{\pi^2}{x}\right).$$

This result is an immediate consequence of the Entry 11 in *Ramanujan's Notebooks* [2, pag. 258]:

*Let  $\alpha, \beta > 0$  with  $\alpha\beta = \pi$ , and let  $n$  be real with  $|n| < \beta/2$ . Then*

$$\alpha \left\{ \frac{1}{4} \sec(\alpha n) + \sum_{k=1}^{\infty} \chi(k) \frac{\cos(\alpha nk)}{e^{\alpha^2 k} - 1} \right\} = \beta \left\{ \frac{1}{4} + \frac{1}{2} \sum_{k=1}^{\infty} \frac{\cosh(2\beta nk)}{\cosh(\beta^2 k)} \right\}$$

*where*

$$\chi(k) = \begin{cases} 0 & \text{for } k \text{ even} \\ 1 & \text{for } k \equiv 1 \pmod{4} \\ -1 & \text{for } k \equiv 3 \pmod{4}. \end{cases}$$

In the Appendix, by making use of the Poisson summation formula and a remarkable Fourier cosine transform, we give a proof of the infinite series identity, which in the author's view is simpler than those given in [2] by Berndt.

For  $n = 0$  this reads

$$(16) \quad \alpha \left\{ \frac{1}{4} + \sum_{k=1}^{\infty} \chi(k) \frac{1}{e^{\alpha^2 k} - 1} \right\} = \beta \left\{ \frac{1}{4} + \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{\cosh(\beta^2 k)} \right\}.$$

Replacing  $\cosh(x)$  by the exponential functions, expanding the geometric series and rearranging the sums we obtain

$$(17) \quad \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{\cosh(\beta^2 k)} = \sum_{m=1}^{\infty} \chi(m) \frac{1}{e^{\beta^2 m} - 1}.$$

Now considering the definition of the harmonic sum  $H(x)$ , we have

$$(18) \quad H(x) = \sum_{m=0}^{\infty} (-1)^m \sum_{n=1}^{\infty} e^{-n(2m+1)x} = \sum_{m=0}^{\infty} \chi(m) \frac{1}{e^{mx} - 1}.$$

So that, taking into account the constraint  $\alpha\beta = \pi$ , we find with (16), (17) and (18):

$$\alpha \left\{ \frac{1}{4} + H(\alpha^2) \right\} = \beta \left\{ \frac{1}{4} + H(\beta^2) \right\}.$$

Finally, we substitute  $\alpha = \sqrt{x}$ ,  $\beta = \pi/\sqrt{x}$  in the above relation and obtain the desired transformation (15).

Plugging this back into our integral (14), we get

$$\begin{aligned} A(s) &= \frac{1}{\pi^s} \int_0^{\infty} H(x) x^{s-1} dx \\ &= \frac{1}{\pi^s} \int_{\pi}^{\infty} H(x) x^{s-1} dx + \frac{1}{\pi^s} \int_0^{\pi} H(x) x^{s-1} dx \\ &= \frac{1}{\pi^s} \int_{\pi}^{\infty} H(x) x^{s-1} dx + \frac{1}{\pi^s} \int_0^{\pi} \left\{ \frac{\pi}{4x} - \frac{1}{4} + \frac{\pi}{x} H\left(\frac{\pi^2}{x}\right) \right\} x^{s-1} dx \\ &= \frac{1}{\pi^s} \int_{\pi}^{\infty} H(x) x^{s-1} dx + \frac{1}{4} \frac{1}{s(s-1)} + \frac{1}{\pi^s} \int_{\pi}^{\infty} \frac{\pi^{2s-1}}{u^s} H(u) du \quad \left(u = \frac{\pi^2}{x}\right) \\ &= +\frac{1}{4} \frac{1}{s(s-1)} + \frac{1}{\pi^s} \int_{\pi}^{\infty} H(x) x^{s-1} dx + \frac{1}{\pi^{1-s}} \int_{\pi}^{\infty} H(x) x^{-s} dx. \end{aligned}$$

Now the whole expression is symmetrical under  $s \mapsto 1-s$ , the integrals on the right hand side define a holomorphic function for all  $s \in \mathbf{C}$ , and so (5) follows.

We summarize this results also in the following theorem:

**Theorem 1:** *The Function*

$$\zeta(s) = \frac{\pi^s}{\Gamma(s) L(s)} \left\{ \frac{1}{4s(s-1)} + \int_{\pi}^{\infty} H(x) \left[ \left(\frac{x}{\pi}\right)^s + \left(\frac{x}{\pi}\right)^{1-s} \right] d \log x \right\}$$

*is meromorphic with a simple pole at  $s = 1$  with residue 1.*

Here is the computation of residue for  $s = 1$  is [ observe that  $L(1) = \arctg(1) = \pi/4$  - a direct consequence of the infinite series expansion of the arctangent (the Madhava-Gregory series)]:

$$\lim_{s \rightarrow 1} (s-1) \zeta(s) = \frac{\pi}{\Gamma(1) L(1)} \cdot \frac{1}{4} = 1.$$

## 6. CONCLUSION

Riemann gave two proofs of the functional equation (and functional identity) in his ground-breaking paper [14], the former argument essentially consists in proving the meromorphic continuation of the  $\zeta(s)$  function and uses contour integration, while the latter, conceptually more difficult, using the  $\theta_3(0|ix)$  theta function, requires the Mellin transformation.

In particular Riemann obtains the symmetrical formulation of the functional identity by using two of the basic identities of the Gamma function, that are: Euler's reflection (or complement) formula and Legendre's duplication formula [13], which was discovered in 1809 and was surely unknown to Euler:

$$\Gamma(s) \Gamma\left(s + \frac{1}{2}\right) = \frac{\sqrt{s}}{2^{2s-1}} \Gamma(2s).$$

Here we have introduced the complex  $A(s)$  function and we have established another symmetrical formulation of the functional equation for the Riemann Zeta function by using the reflection formulae of the  $\zeta(s)$ ,  $L(s)$  and  $\Gamma(s)$  functions, all well-known by Euler.

Using the definition of the  $A(s)$  function we are also able to obtain several integral representations of  $A(s)$  (with  $\Re(s) > 0$  or  $\Re(s) > 1$ ), that connect in an amazing way the  $\zeta(s)$  function with the independent transcendent  $L(s)$  function.

In addition, as a direct consequence of the second integral representation, we have obtained an analytic continuation of the same function by the Mellin transform of a function, defined by an infinite harmonic sum, and using an identity of Ramanujan.

This last result represents another proof of the functional equation for the  $A(s)$  function, that is independent of the three reflection formulae of the  $\zeta(s)$ ,  $L(s)$  and  $\Gamma(s)$  functions.

Finally, it is possible to state the following theorem:

**Theorem 2:** *The  $A(s)$  function extends itself as a meromorphic function in the complex field  $\mathbf{C}$ , in a regular way, except for the simple poles at  $s = 0, 1$  [ respectively determined by the Gamma function  $\Gamma(s)$  and by the Zeta function  $\zeta(s)$  ] and satisfies the remarkable functional equation:*

$$A(s) = A(1-s).$$

*As the complex zeros of the  $A(s)$  function coincide with the nontrivial zeros of the  $\zeta(s)$  and  $L(s)$  functions, they are localized in the strip, determined by  $0 \leq \Re(s) \leq 1$ .*

Let us keep in mind that in the  $A(s)$  function the singularities of the  $\Gamma(s)$  function, that we can find in the negative real axis are cancelled by trivial zeros of the two Euler's Zeta functions  $\zeta(s)$  and  $L(s)$ .

It is in fact immediate to verify, from each of the functional equations (1) and (2), exploiting the zeros of the trigonometric functions *cosine* and *sine*, that:

$$\zeta(s) = 0 \quad \text{for } s = -2, -4, -6, -8, \dots \quad \text{and} \quad L(s) = 0 \quad \text{for } s = -1, -3, -5, -7, \dots$$

Since the Gamma function has no zeros, and since the Dirichlet Beta function and the Riemann Zeta function have an Euler product (see §1. Introduction), which shows that both are nonvanishing in the right half plane  $\Re(s) > 1$ , the function  $A(s) = \Gamma(s) \pi^{-s} \zeta(s) L(s)$  has no zeros in  $\Re(s) > 1$ .

By the functional equation  $A(s) = A(1-s)$ , it also has no zeros in  $\Re(s) < 0$ : thus all the zeros have their real parts between 0 and 1.

Moreover, if all the complex zeros of the function  $A(s)$  have their real part equal to  $1/2$ , we shall obtain, as results, both a proof of the Riemann Hypothesis, and the following assertion [9] of Tschebyschef:

*The function* 
$$F(y) = e^{-3y} - e^{-5y} + e^{-7y} + e^{-11y} - \dots = \sum_{p>2} (-1)^{\frac{p+1}{2}} e^{-py}$$
  
*tends to infinity, as  $y \rightarrow 0$*  (the summation variable  $p$  is to run over all odd primes).

Indeed, in the paper [9] Hardy and Littlewood prove that the statement made by Tschebyschef is true if all complex zeros of the function  $L(s)$  have their real part equal to  $1/2$ .

This result confirms, in a very subtle way, the preponderance of primes of the form  $4m+3$  [16, pag. 125].

Roughly speaking, there are "more" primes congruent to 3 mod 4 than congruent to 1 mod 4.

The historical memoir of Riemann on the Zeta function has been naturally extended to the family of Dirichlet L-functions including the Riemann hypothesis.

The so-called Grand Riemann Hypothesis asserts that all the zeros of  $L(s, \chi)$  in the critical strip  $0 < \Re(s) < 1$  are on the critical line  $\Re(s) = \frac{1}{2}$  and this is a good reason to finish the paper right here.

## 7. APPENDIX

We prove the following infinite series identity of Ramanujan ( written in the inverse order):

$$(19) \quad \beta \left\{ \frac{1}{4} + \sum_{k=1}^{\infty} \frac{\cosh(2\beta nk)}{\cosh(\beta^2 k)} \right\} = \alpha \left\{ \frac{1}{4} \sec(\alpha n) + \sum_{k=1}^{\infty} \chi(k) \frac{\cos(\alpha nk)}{e^{\alpha^2 k} - 1} \right\}$$

with  $\alpha, \beta > 0$ ,  $\alpha\beta = \pi$  and  $n$  real with  $|n| < \beta/2$ , using the Poisson summation formula:

$$(20) \quad \frac{1}{2}f(0) + \sum_{k=1}^{\infty} f(k) = \int_0^{\infty} f(x) dx + 2 \sum_{k=1}^{\infty} \int_0^{\infty} f(x) \cos(2k\pi x) dx$$

and the following Fourier cosine transform [3, cap. VII, pag.174, 7.112] with  $0 < a < b$ :

$$(21) \quad F_c(u) = \int_0^{\infty} \frac{\cosh(at)}{\cosh(bt)} \cos(ut) dt = \frac{\pi}{b} \frac{\cos\left(\frac{\pi a}{2b}\right) \cosh\left(\frac{\pi u}{2b}\right)}{\cos\left(\frac{\pi a}{b}\right) + \cosh\left(\frac{\pi u}{b}\right)}.$$

We take into (20):

$$f(x) = \frac{\cosh(2\beta nx)}{\cosh(\beta^2 x)}$$

and considering (21) with  $a = 2\beta n$ ,  $b = \beta^2$ ,  $u = 2k\pi$  we have then:

$$\int_0^{\infty} \frac{\cosh(2\beta nx)}{\cosh(\beta^2 x)} \cos(2\pi kx) dx = \frac{\alpha}{\beta} \frac{\cos(n\alpha) \cosh(k\alpha^2)}{\cos(2n\alpha) + \cosh(2k\alpha^2)}.$$

By the relation  $\cos z = \cosh(iz)$  and the following prostapheresis-formula:

$$\cosh(p) + \cosh(q) = 2 \cosh\left(\frac{p+q}{2}\right) \cdot \cosh\left(\frac{p-q}{2}\right)$$

we have immediately:

$$(22) \quad \int_0^{\infty} \frac{\cosh(2\beta nx)}{\cosh(\beta^2 x)} \cos(2\pi kx) dx = \frac{\alpha}{4\beta} \left[ \operatorname{sech}(k\alpha^2 - in\alpha) + \operatorname{sech}(k\alpha^2 + in\alpha) \right].$$

Observe that setting  $k = 0$  we have:

$$(23) \quad \int_0^{\infty} \frac{\cosh(2\beta n x)}{\cosh(\beta^2 x)} dx = \frac{\alpha}{2\beta} \sec(n\alpha).$$

In short considering:

$$\frac{1}{\cosh(x)} = \operatorname{sech}(x) = 2 \sum_{r=1}^{\infty} \chi(r) e^{-rx}$$

and besides:

$$\left[ \operatorname{sech}(k\alpha^2 - in\alpha) + \operatorname{sech}(k\alpha^2 + in\alpha) \right] = 4 \sum_{r=1}^{\infty} \chi(r) e^{-2k\alpha^2} \cos(\alpha n r)$$

we also deduce that:

$$\sum_{k=1}^{\infty} \left[ \operatorname{sech}(k\alpha^2 - in\alpha) + \operatorname{sech}(k\alpha^2 + in\alpha) \right] = 4 \sum_{r=1}^{\infty} \chi(r) \frac{\cos(\alpha n r)}{e^{\alpha^2 r} - 1}.$$

As a consequence of (22) putting the above equality with (23) into (20) we have the identity (19).

## 8. ADDITIONAL REMARKS

*Remark 1.* The functional equation (5) for  $A(s)$  is not new.

In [L. Lorenz, *Tidskr. Mat.* **1**, 97 (1871)] it is shown that:

$$A(s) = \frac{\Gamma(s)}{4\pi^s} Z \left| \begin{array}{c} \vec{0} \\ \vec{0} \end{array} \right| (1; 2s)$$

in terms of Epstein's Zeta-function:

$$Z(1; 2s) = \sum_{\substack{m, n = -\infty \\ (m, n) \neq (0, 0)}}^{\infty} \frac{1}{(m^2 + n^2)^s}$$

[P. Epstein, *Zur Theorie allgemeiner Zetafunktionen. I.*, Math. Ann. **56**, 615 (1903)].

This function satisfies the functional equation:

$$\pi^{-s} \Gamma(s) Z(1; 2s) = \pi^{-(1-s)} \Gamma(1-s) Z(1; 2-2s)$$

which is the same as  $A(s) = A(1-s)$ .



*Remark 2.* One motivation for the surprisingly quick proof of the symmetrical formulation [see (5) or (8)] of the functional equation of the Zeta function is that Euler himself could have proved this remarkable identity with the three reflection formulae (this is the reason for a dedication to Leonhard Euler).

*Remark 3.* We could have obtained an analytic continuation of the function  $A(s)$  also from the first integral representation (11), following a similar method (the transformation law of theta function ) to the one used by Riemann, but we have chosen a second opportunity, just to give a more innovative proof with the identity of Ramanujan and therefore slightly different from the classical one, that we find in Riemann's original memoir.

**Dedication.** This paper is written honor of **Leonhard Euler**, one of the greatest mathematicians and mechanicians of all time.



**LEONHARD EULER (1707-1783)**

(from the portrait by A. Lorgna)

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